Transmission-Line Properties of a Round Wire in a Polygon Shield

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Abstract—**A family of transmission lines is based on a round wire in a cylindrical shield of polygon cross-section. There is presented a simple formula which gives a smooth transition between the extremes of a small wire and a wire near contact. The same formula is adapted for different shapes by entry of different constants depending on the number of shield planes and the degree of symmetry. The formula is reversible for synthesis or analysis by explicit expression of either shape ratio or wave resistance in terms of the other. For comparison in the transition region, there is computed for each shape a close approximation by the method of images.**

I. Introduction

A FAMILY of simple transmission lines is based on a round wire in a cylindrical shield of polygon cross-section. An elementary formula for each shape has long been available for a wire much smaller than the space in the shield [1] [4]. The author published in 1950 [1] a set of curves showing the behavior for a larger wire, approaching contact. That set was based on an orderly transition between the extremes of a small wire and one near contact. In the meantime, further studies have yielded more understanding and theoretical tools for formulating the transition behavior [2] [3] [5] [8]. There is here presented a general formula which gives remarkably close approximation to the known behavior in a variety of shapes. This formula is reversible, in that it yields for synthesis or analysis a simple explicit solution for either the shape ratio or the wave resistance in terms of the other.

II. Symbols

MKS rationalized units (meters, ohms, etc.)

- R = wave resistance of the transmission line formed by the wire and the shield (so-called "characteristic impedance") with free space as the dielectric.
- d = diameter of the wire (inner conductor).
- D = diameter of a wire that would contact the shield (all planes).
- h = $\frac{1}{2}(D-d)$ = height of one wire above one plane (or separation of the wire from every plane).
- S = D/d = diameter ratio (shape ratio).
- s = shield ratio by which the effective diameter (for a small wire) exceeds the contact diameter (D).
- n = number of planes forming the shield.
- m = exponent in the general formula.
- c = $2m/n^2$ = a constant in the general formula.
- k = dielectric constant of sheet of material separating the wire from the one plane.
- $\ln x$ = $\log_e x$.
- $\exp x$ = e^x.
- acosh x = anticosh x = $\cosh^{-1} x$.

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III. A Round Wire in a Polygon Shield

Fig. 1 illustrates the symbols and dimensions used herein, with reference to a single plane as a one-sided polygon shield. This is identified as shape (1). The shape ratio ($S = D/d$) is increased by the shield ratio ($s = 2$) which is a constant in the limit of a large ratio ($S \gg 1$) peculiar to a small wire. The corresponding effective outer diameter (sD) is shown by a dashed line. The separation from contact is denoted the height (h) which is here again used as a reference in the "normalized power factor" [2] [3].

Any shape of line may be filled with dielectric (k), in which case its wave resistance (R) is decreased by the familiar factor ($1/\sqrt{k}$). This is omitted herein.

There is one case of dielectric which is interesting and may be particularly useful. It is identified as shape ($1k$) in Fig. 1. The round wire is separated from a shield plane by a sheet of dielectric (k). This may be useful on a printed-circuit board for reducing the loss as compared with a printed strip [14].

Here the effective dielectric constant (k') is formulated simply:

$$k' = \sqrt{k}. \tag{1}$$

This rule has been derived for these two extreme cases:

(a) Any shape ratio, with weak dielectric ($k - 1 \ll 1$).
(b) Any dielectric, with wire near contact ($S - 1 \ll 1$) [15].

It is taken as a fair approximation for all cases, and may even have an exact basis without restriction.

Fig. 2 shows the six shapes that are evaluated herein. The first and last are susceptible of exact evaluation by known reversible formulas. The intermediate four exemplify the need for the general formula here presented. Each shape is associated with these numbers or values:

- (1–6) The identity of the shape of the shield.
- (s) The shield ratio [1] [4].
- (n) The number of planes forming the polygon shield.
- (m) An exponent used in the formula.
- (c) A derived constant used in the formula ($c = 2m/n^2$).

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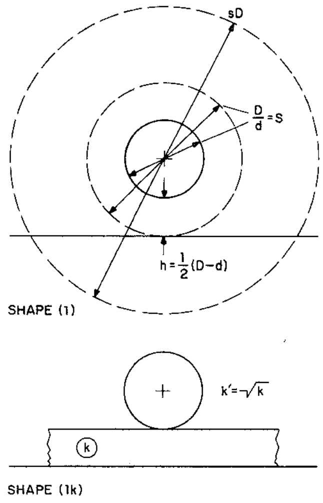

Fig. 1. Dimensions of a round wire near a plane.

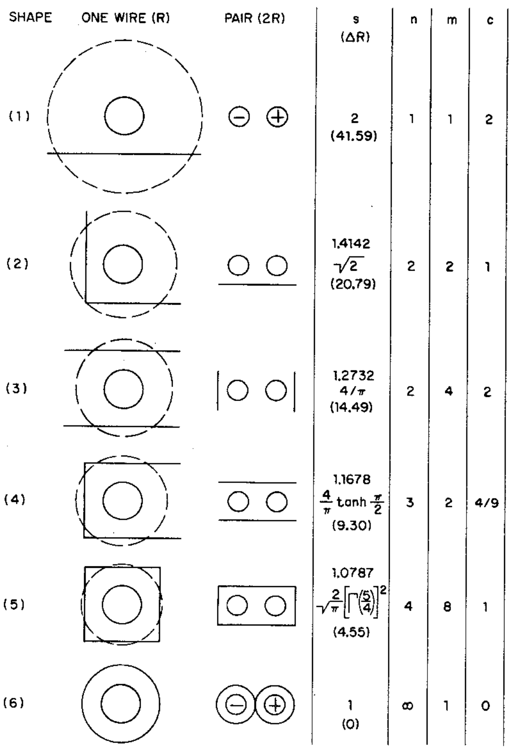

Fig. 2. Cross-section shapes and constants.

Also shown is the increment (ΔR) by which the shield ratio increases the wave resistance for a small wire:

$$\Delta R = 60 \ln s. \tag{2}$$

Each of the shapes of shield may be described as follows [1]:

(1) One plane ($n=1$).
(2) A corner ($n=2$).
(3) Parallel planes ($n=2$).
(4) Channel ($n=3$).
(5) Square cylinder ($n=4$).
(6) Circular cylinder ($n=\infty$).

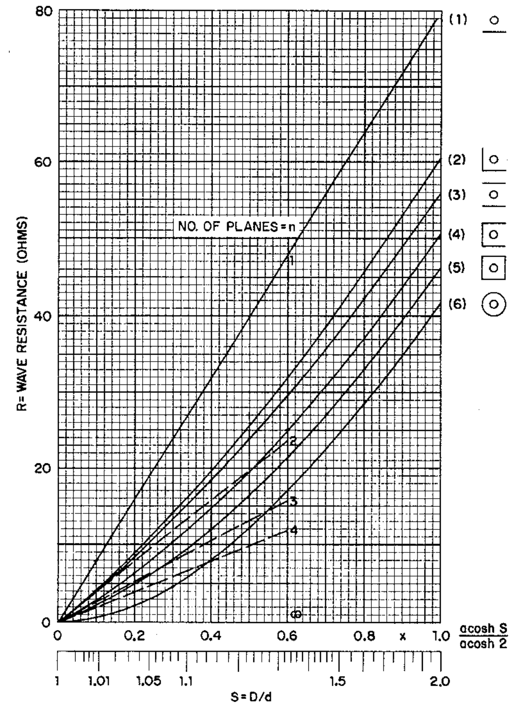

Fig. 3. Curves of wave resistance.

The wave resistance (R) is defined for one wire in such a shield. The corresponding balanced pair ($2R$) is also shown.

Shape (4) is particularly well suited for use as a slotted line, because the open side can be located at the top for continuous access. The side walls can be made wide enough for any degree of shielding. Then the walls provide a track for the traveling probe. In contrast to shape (3) commonly used for this purpose, the third wall prevents propagation of the TEM extra mode between the parallel walls.

In 1945, at the Hazeltine laboratory in Little Neck, a long slotted line was needed for FM and TV bands. A length of 12 ft provided a half-wave down to 40 MHz. One line was built with shape (4), about 3-in wide, the inner conductor being supported by a thin dielectric fin above the bottom wall. Another line was built with a balanced pair of inner conductors in a single trough [1]. It was equivalent to two single lines side-by-side with the center partition removed. The balanced magnetic field was probed by a horizontal loop midway between the walls. The single line was the predecessor of the shape (3) slotted lines which came into common use a few years later [5].

Fig. 3 shows the wave resistance in the transition range of shape ratio. This was first published in 1950 [1] in a similar form, but without a formula. This feature expands the region near contact. As a further refinement here, the scale of abscissas is chosen to give a straight line for the top graph (1). The lower graphs have a lesser slope ($1/n$) near the origin but approach the same slope far from the origin (small wire).

IV. The Reversible General Formula

The general formula is here stated for explicit synthesis and analysis.

$$S = D/d = \left\{ 1 + \frac{(\exp mR/60 - 1)^2}{s^m(\exp mR/60 - 1) + c} \right\}^{1/m} \quad (3)$$

$$R = \frac{60}{m} \ln \left\{ 1 + \left[\frac{1}{2} s^m (S^m - 1) \right] + \sqrt{\left[\frac{1}{2} s^m (S^m - 1) \right]^2 + c(S^m - 1)} \right\}. \quad (4)$$

The constants (s, m, c) are given in Fig. 2 for each shape.

The top and bottom shapes have known reversible explicit formulas, to which the general formula simplifies with the listed constants. The wire near one plane, shape (1), is based on its elementary form:

$$S = D/d = \cosh R/60; \qquad R = 60 \operatorname{acosh} D/d. \quad (5)$$

The coaxial circular shape (6) is best expressed in its elementary form:

$$S = D/d = \exp R/60; \qquad R = 60 \ln D/d. \quad (6)$$

The wire in a corner, shape (2), has some peculiarities which led to the form of the general formula. In Fig. 3, its R is the average of the top and bottom graphs at both extremes of shape. One might infer that the average would be a close approximation in the transition region (though not exact). Expressing this average leads to the form of the general formula with the listed constants.

It was perceived that this form could be provided with a different set of constants, as listed, to give the exact formulas for the top and bottom extreme shapes.

For any number of planes, the known constants (s, n) give the correct slopes in Fig. 3 at both extremes of shape. For a small wire:

$$R = 60 \ln sD/d; \qquad S = D/d = \frac{1}{s} \exp R/60. \quad (7)$$

Near contact:

$$S = D/d = 1 + \frac{1}{2}(nR/60)^2; \qquad R = \frac{60}{n}\sqrt{2(S-1)}\,. \quad (8)$$

There remains a choice of one constant (m, giving c) to enable the general formula to give closest approximation in the transition region between the extremes. This choice for any one shape would be based on whatever knowledge is available.

For a wire between parallel planes, shape (3), intensive studies had been made in the meantime [6] [8]. These had placed R within rather close bounds for all shape ratios and had yielded close explicit approximations for synthesis and analysis. Considerations of symmetry and the second approximation for a small wire lead to one choice of constants ($m = 4$, $c = 2$) This choice gives a close approximation to the closest known transition. Also this result is found to be only weakly dependent on the choice.

TABLE I
COMPARISON

Shape	R for d/D = 0.95		
	Close Approx.	General Formula	Relative Error
(1)	19.382	19.382	0
(2)	11.132	11.230	+.0088
(3)	10.646	10.676	+.0028
(4)	7.886	7.982	+.0123
(5)	6.261	6.303	+.0067
(6)	3.078	3.078	0

The general formula with these constants contains these features:

a) For a small wire, the correct first term, and a second term which is correct in form ($m = 4$) and approximately in amount.
b) For a wire near contact, the correct first term, and a second term which appears to track approximately.

For a wire in a square cylinder, shape (5), similar considerations made it logical to double the exponent ($m = 8$, $c = 1$).

For a wire in a channel, shape (4), there is a lower order of symmetry, comparable with shape (3), so the same exponent is chosen ($m = 2$, $c = 4/9$).

It is noted that the stated choice of the exponent (m, c) for each number of planes is that which gives nearly the least R in the transition region. This corresponds to least departure from the straight lines having the slopes established for both extremes of shape ratio. The result is found to be only weakly dependent on this choice.

The general formula is seen to be exact for all shape ratios in the two extreme shapes, while leaving some amount of uncertainty in the transition region for the intermediate shapes. For estimating the residual error of R, each intermediate shape has been computed very closely for one or more points in the region of most uncertainty. This computation is based on images in the shield planes. It is described in the Appendix, which gives the computed values for testing the general formula.

Table I is a summary of values obtained by close approximation and by the general formula. For each shape, the relative error of the formula is near the maximum for all ratios. It is near one percent for most of the intermediate shapes. However, it is much less (<0.3 percent) for parallel planes, the case that has received most attention. [5] [6] [8] The greatest error occurs for rather low values of R, which are more sensitive to dimensional tolerances and are seldom used.

V. The Loss Power Factor

An earlier paper [2] describes a general method for computing the magnetic-loss power factor (PF) caused by the skin effect. That method uses numerical differentiation of wave resistance by increment of dimensions. That method is equally applicable here. It yields the actual loss PF and also the normalized loss PF by reference to the separation (h).

For a pair of wires, the loss PF may be decreased by the removal of all or part of the shield, leaving a plane of symmetry free of loss. To simulate this reduction of loss, the increment of D is multiplied by the fraction:

$$\frac{\text{no. of remaining shield planes around each wire}}{\text{basic no. of shield planes around one wire}}. \quad (9)$$

The normalized loss PF of shapes (1) and (3) is graphed in [2] and [3].

In an earlier paper [7] the author gives a simple rule for the skin resistance of a polygon shield around a small wire. It is equal to the resistance of the inscribed circular shield (like the outer conductor of the simple coaxial line).

VI. Conclusion

For the subject family of lines, a simple reversible formula is available for synthesis or analysis of any shape ratio between the extremes of a small wire and a wire near contact.

Appendix
Close Approximation by Images

The round wire in a polygon shield offers an opportunity for close approximation by the method of images. The limiting case of a small wire yields a simple reversible formula based on its images in the plane walls of the polygon shield. This has long been known [1] [4] [9]. The approximation has been close enough for practical sizes of wire not too close to contact with the shield.

For either extreme case of a shield made of one or many planes, a reversible formula has been known for any shape ratio out to contact. Among the intermediate shapes, only the wire between parallel planes (3) has received intensive analysis [5] [6] [8] and no simple formula for analysis or synthesis has previously emerged.

There are analytic approaches which have general application but are so laborious as to require much computer capacity.

The method of images can be extended for approximate computation of a round wire of any shape ratio, even approaching contact, if the shield planes form one or more sides of a regular polygon. Not all sides need be represented. The intermediate shapes here considered meet this condition. The reference polygon is a square circumscribed on the outer diameter (D).

The method used here is shown in Fig. 4 for one shape, a round wire in a corner. A pattern of lines and images is shown as an example. The same approach has been shown for parallel planes [6] [8]. Any finite pattern cannot realize

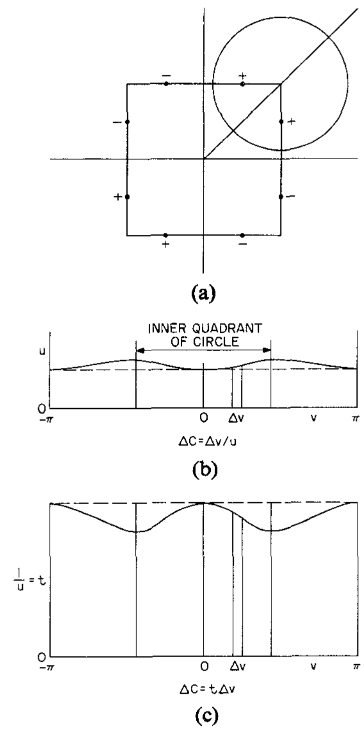

Fig. 4. Close approximation for a wire in a corner. (a) The pattern of line images. (b) The graph of potential over flux. (c) The area of inverse potential times flux.

a locus of constant potential on the wire circle. However, the image pattern can be designed to reduce the variation of potential on the circle. The variation can be made small enough to be accommodated in an averaging process.

The problem is most severe for a wire near contact. With a single plane, one line and its image can be located at a conjugate pair of "focal" points to give constant potential on the plane (zero) and on any given circle near the plane. For each plane, a similar pair of images is provided, as shown in Fig. 4(a). This is found to assure that the potential variation around the circle has a ratio of variation not much different from unity. In the case shown, this ratio is less than 4/3. Also it happens that equal maxima of potential occur at the nearest and furthest points on the diagonal line of symmetry.

Fig. 4(b) shows the variation of potential (u) around the circle. It is graphed on a scale of flux (v). Naturally the flux is concentrated in the regions near contact, where the potential is near minimum.

The concept of capacitance is helpful in summing the effects around the circle. Any small element of flux (Δv) is associated with a value of potential (u) to contribute an element of capacitance ($\Delta C = \Delta v/u$). If the potential and flux lines depart only slightly from a rectangular grid, the sum of capacitance from this graph is a close approximation for the circular wire. This condition is met for a low graph, as shown. Also it would be met for a high graph, which would have a much smaller ratio of variation.

Fig. 4(c) shows a graph of inverse potential ($t = 1/u$). On this graph, an element of area is an element of capacitance ($\Delta C = t\Delta v$). This area can be integrated numerically to give the capacitance (C), which determines the wave resistance ($R = 377/C$).

This algorithm amounts to averaging around the circle by a particular rule. It is a refinement of the simple averaging previously used for parallel planes [6] [8].

Any pattern of images can be relocated to reduce the variation of potential around the circle. The image point can be shifted in two dimensions so two conditions can be met. In Fig. 4, for example, the images can be adjusted to double the number of ripples between upper and lower bounds, which places these bounds much closer. For the shape ratio in Table I, the ratio of the bounds can be reduced from 1.26 to 1.10, thereby decreasing any uncertainty in the approximation involved in the algorithm.

If the area is integrated in steps around the circle, the flux increments (Δv) are not uniform. The trapezoid rule does not require equal increments, so it is suitable for this purpose. For any one quadrant of the circle, less than 20 steps were required for close approximation in the cases here reported.

As a convenience for numerical integration and interpretation, the angle around the circle and the flux around the line images may be expressed in circle units or in grads (400 = 1 circle).

The computation of potential and flux (u, v) around the circle is simple for all shapes here considered. This and the numerical integration are well within the capability of a programmable personal calculator with printout (such as the HP-97 used by the author).

In some cases, the images may be located further from the center of the circle, to realize some advantage. This was done for parallel planes in order to place the potential variation within closest bounds [6] [8]. Critical location of the images offered a great advantage in that case. For an intermediate range of shape ratio, the averaging over the circle was placed within rather close bounds.

A second approximation for a small wire can be obtained with images at the center of the circle, the simplest pattern.

There may be noted, some interesting relationships among the shapes shown in Fig. 2.

As mentioned, shape (2) is evaluated by the average R of shapes (1) and (6) at both extremes of the shape ratio. It is evaluated for any ratio by the image pattern in Fig. 4(a). Table I, for one ratio, shows the average to differ from the close approximation by a small relative error (0.0060). Therefore, the rule is not exact, though helpful. The error approaches zero toward either extreme of shape ratio.

Shape (4) is between (3) and (5). Its space can be made up of one half of the square (5) and one half of the parallel planes (3). The field lines do not match but the departure is small enough to make this concept useful. The resulting rule is: the R of (4) is the harmonic mean of (3) and (5). The error of this rule is only 0.2 Ω in the limit of high R and less for lower R, approaching zero relative error near contact.

If a wire in a square is near contact, as in Table I, shape (6), its computation is greatly simplified by taking the inner quadrant of the circle in Fig. 4(a). The entire circle has 4 times this C. Otherwise the square would require a doubly infinite set of images (which could be summed in closed form in terms of elliptic functions).

This algorithm for a close approximation has proved to be useful for a round wire in a polygon shield. The "focal" location of images is good for all shapes and ratios. It is simple to locate. Some further refinement is usually realized by locating the images a little further from the center of the circle.

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